Nonlinear Partial Differential Equations Solved by Projective Riccati Equations Ansatz

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Based on the general projective Riccati equations method and symbolic computation, some new exact travelling wave solutions are obtained for a nonlinear reaction-diffusion equation, the high-order modified Boussinesq equation and the variant Boussinesq equation. The obtained solutions contain solitary waves, singular solitary waves, periodic and rational solutions. From our results, we can not only recover the known solitary wave solutions of these equations found by existing various tanh methods and other sophisticated methods, but also obtain some new and more general travelling wave solutions.

Key words: General Projective Riccati Equations Method; Nonlinear Partial Differential Equation; Symbolic Computation; Travelling Wave Solution.

1. Introduction

Finding exact solutions, in particular, solitary wave solutions, of nonlinear partial differential equations (NPDEs) arising from many science fields is of important significance. Up to now, there have been a wealth of methods for finding exact solutions of NPDEs, such as, the inverse scattering transform method [1, 2], Bäcklund transformation [1 – 3], bilinear method [4], various tanh methods [5 – 9], separation of variable method [10], generalized hyperbolic-function method [11] and general Riccati equations expansion method [12] and so on. Particularly, with the rapid development of computerized symbolic computation, some general ansatz method have been proposed in order to obtained new formal solutions for given NPDEs.

In 1992, Conte and Musette [13] presented an indirect method to seek more solitary wave solutions of some NPDEs that can be expressed as polynomials in two elementary functions which satisfy a projective Riccati system [14]. Using this method, many solitary wave solutions of many NPDEs are found [14, 15]. Recently, Yan [16] developed further Conte and Musette's method by introducing more general projective Riccati equations.

In this paper, making use of the general projective Riccati equations method and symbolic computation—*Maple*, we consider a nonlinear reaction-diffusion equation [17, 18], the high-order modified Boussinesq equation [18, 19] and the variant Boussinesq equation [20 – 22].

The paper is organized as follows. In Sect. 2, we summarize the general projective Riccati equations method. In Sect. 3, making use of the method and symbolic computation, many families of exact solutions for a nonlinear reaction-diffusion equation, the high-order modified Boussinesq equation and the variant Boussinesq equation are found.

2. General Projective Riccati Equations Method

The key idea of the method is to take full advantages of the following subequations, namely the projective equations [14 – 16]:

$$\begin{aligned}
&\frac{\mathrm{d}\sigma(\xi)}{\mathrm{d}\xi} = \varepsilon\sigma(\xi)\tau(\xi),\\
&\frac{\mathrm{d}\tau(\xi)}{\mathrm{d}\xi} = R + \varepsilon\tau^2(\xi) - \mu\sigma(\xi),\\
&\varepsilon = \pm 1,
\end{aligned} \tag{1}$$

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where R, μ are constants. It is easy to see that (1) admits the first integral with $R \neq 0$,

$$\tau^2(\xi) = -\varepsilon[R - 2\mu\sigma(\xi) + \frac{\mu^2-1}{R}\sigma^2(\xi)]. \quad (2)$$

We know that (1) admits the following solutions:
Case 1. When $\varepsilon = -1, R \neq 0$,

$$\begin{aligned} \sigma_1(\xi) &= \frac{R\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}, \\ \tau_1(\xi) &= \frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}, \\ \sigma_2(\xi) &= \frac{R\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}, \\ \tau_2(\xi) &= \frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}. \end{aligned} \quad (3)$$

Case 2. When $\varepsilon = 1, R \neq 0$,

$$\begin{aligned} \sigma_3(\xi) &= \frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1}, \\ \tau_3(\xi) &= \frac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi)+1}, \\ \sigma_4(\xi) &= \frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}, \\ \tau_4(\xi) &= -\frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi)+1}. \end{aligned} \quad (4)$$

Case 3. When $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{C}{\xi} = C\varepsilon\tau_5(\xi), \quad \tau_5(\xi) = \frac{1}{\varepsilon\xi}, \quad (5)$$

where C is a constant.

Now we describe the general projective Riccati equations method as follows.

Given a NPDE with, say, two variables: $\{x, t\}$,

$$p(u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \cdots) = 0. \quad (6)$$

Under the transformation $u(x,t) = u(\xi)$, $\xi = x - \lambda t$, (6) reduces to

$$G(u', u'', u''', \cdots) = 0. \quad (7)$$

Step 1. By Balancing the highest order derivative term and the nonlinear terms in (7), we can find the balance constant m (m is usually a positive integer). If m is a fraction or a negative integer, we first make the transformation

$$u(\xi) = \varphi^m(\xi), \quad (8)$$

then substitute (8) into (7) and return to determine balance constant m again.

Step 2. We assume that (7) has the following two solutions:

Type 1. When $R \neq 0$,

$$u(\xi) = A_0 + \sum_{i=1}^{m} \sigma^{i-1}[A_i\sigma(\xi) + B_i\tau(\xi)], \quad (9)$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy Eqs. (1)–(2).

Type 2. When $R = \mu = 0$,

$$u(\xi) = \sum_{i=0}^{m} A_i\tau^i(\xi), \quad (10)$$

where $\tau'(\xi) = \tau^2(\xi)$.

Step 3. Substituting (1), (2) and (9) (or (10)) into (7), yields a set of algebraic polynomials for $\sigma^j(\xi)\tau^i(\xi) (j = 0, 1, \cdots; i = 0, 1)$ (or $\tau^l(\xi), l = 0, 1, \cdots$). Setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ (or $\tau^l(\xi)$) to zero yields a set of over-determined algebraic polynomials of λ, A_i, B_i, R and μ.

Step 4. Using the symbolic computation system – *Maple*, solving the above set of algebraic equations, yields the values of A_i, B_i, R, λ.

Thus according to (3)–(5), (8)–(10) and the conclusions in *Step 4*, we can obtain many families of exact travelling wave solutions for (6).

3. Applications to Some Nonlinear Partial Differential Equations

3.1. A Nonlinear Reaction-diffusion Equation

Let us consider a nonlinear reaction-diffusion equation [17, 18]

$$u_t = (au^p u_x)_x - bu + qu^{p+1}, \quad (11)$$

By use of the tanh method, Khater *et al.* [17] found a stationary periodic solutions of (11). In [18], Li *et al.*, obtained four families of exact travelling wave solutions by extended-tanh method.

Let $u(x,t) = v(\xi)$, $\xi = x - \lambda t$, then (11) reduces to

$$a(v^p v')' + \lambda v' - bv + qv^{p+1} = 0. \quad (12)$$

Balancing between $(v^p v')'$ and v' yields $m = -\frac{1}{p}$, which needs not be a positive integer. We make transformation

$$v(\xi) = \phi^{-\frac{1}{p}}(\xi), \tag{13}$$

then (12) becomes

$$-ap\phi\phi'' + a(1+2p)\phi'^2 + p\phi^2(pq - \lambda\phi') - bp^2\phi^3 = 0. \tag{14}$$

Now balancing terms $\phi\phi''$ (or ϕ'^2) and $\phi^2\phi'$ in (14) gives the balancing number $m = 1$. Therefore we assume that

$$\phi = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \tag{15}$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy (1)–(2) and A_0, A_1, B_1 are undetermined constants.

Substituting (1), (2) and (15) into (14), we get a set of algebraic polynomials for $\sigma^j(\xi)\tau^i(\xi)(j = 0, 1, \cdots; \quad i = 0, 1)$. Setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ equal to zero, we obtain an overdetermined set of algebraic polynomial equations with respect to A_0, A_1, B_1, μ and λ. For simplicity, in this paper we only consider the case with $\varepsilon = -1$.

$$-p^2R^2(-RB_1{}^2q + 3RbA_0B_1{}^2 - A_0{}^2q + bA_0{}^3) = 0, \tag{16}$$

$$-B_1p^2R^2(B_1{}^2bR + 3bA_0{}^2 - 2A_0q) = 0, \tag{17}$$

$$\begin{aligned} &A_1R(\lambda\, pRA_1{}^2 + 3\,pB_1{}^2\lambda\,\mu^2 - 2\,aB_1 - 3\,pB_1{}^2\lambda \\ &\quad + 2\,aB_1\mu^2) = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &-R(4\,RpA_1B_1{}^2\lambda\,\mu - 2\,RpA_0A_1{}^2\lambda + 2\,RaB_1A_1\mu \\ &\quad + 3\,Rbp^2A_1{}^2B_1 + bp^2B_1{}^3\mu^2 - bp^2B_1{}^3 \\ &\quad + 2\,pA_0B_1{}^2\lambda - 2\,pA_0B_1{}^2\lambda\,\mu^2 - 2\,apA_0B_1 \\ &\quad + 2\,apA_0B_1\mu^2) = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} &pR^2(-3\,pbA_1B_1{}^2R + 2\,pA_0A_1q - 3\,pbA_0{}^2A_1 \\ &\quad + 6\,pbA_0B_1{}^2\mu - 2\,pB_1{}^2q\mu + 2\,RA_0A_1\lambda\,B_1 \\ &\quad + Ra_1{}^2\mu - RaA_0A_1 - RB_1{}^3\mu\,\lambda - A_0{}^2\lambda\,B_1\mu) = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &-R(2\,RaA_1{}^2\mu + Rbp^2A_1{}^3 + RapA_1{}^2\mu + 5\,RpA_1{}^2\lambda\,B_1\mu \\ &\quad + 4\,pA_0B_1\lambda\,A_1 - 2\,aB_1{}^2\mu - apB_1{}^2\mu^3 - 2\,apA_0A_1 \\ &\quad + 3\,B_1{}^3\mu^3p\lambda - 3\,B_1{}^3\mu\,p\lambda - 4\,pA_0B_1\lambda\,A_1\mu^2 \\ &\quad + 3\,bp^2A_1B_1{}^2\mu^2 - 3\,bp^2A_1B_1{}^2 + 2\,apA_0A_1\mu^2 \\ &\quad + apB_1{}^2\mu + 2\,B_1{}^2\mu^3a) = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &R(R^2aA_1{}^2 + R^2apA_1{}^2 + 2\,R^2pA_1{}^2\lambda\,B_1 + 2\,RapB_1{}^2 \\ &\quad + RaB_1{}^2\mu^2 - 2\,RapB_1{}^2\mu^2 + 3\,RapA_0A_1\mu \\ &\quad - 3\,Rbp^2A_0A_1{}^2 + 3\,RB_1{}^3\mu^2p\lambda + 6\,Rbp^2A_1B_1{}^2\mu \\ &\quad - RB_1{}^3p\lambda - 6\,RpA_0A_1\lambda\,B_1\mu + RA_1{}^2p^2q \\ &\quad + 3\,bp^2A_0B_1{}^2 - 3\,bp^2A_0B_1{}^2\mu^2 + B_1{}^2p^2q\mu^2 \\ &\quad - pA_0{}^2\lambda\,B_1 - B_1{}^2p^2q + pA_0{}^2\lambda\,B_1\mu^2) = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &(\mu-1)(\mu+1)(B_1{}^2\mu^2a + B_1{}^3\mu^2p\lambda - B_1{}^2a - B_1{}^3p\lambda \\ &\quad + RaA_1{}^2 + 3\,RpA_1{}^2B_1\lambda) = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &-pR^2(-2\,pA_1B_1q + 6\,pbA_0A_1B_1 - 2\,pbB_1{}^3\mu \\ &\quad + aB_1A_1R - RA_1B_1{}^2\lambda + 2\,A_0B_1{}^2\lambda\,\mu \\ &\quad - A_0{}^2\lambda\,A_1 - aA_0B_1\mu) = 0. \end{aligned} \tag{24}$$

By use of the *Maple* software package "charsets" by Dongming Wang, which is based on the Wu-elimination method [23–25] which is a sufficient method to solve the systems of algebraic polynomial equations with more unknowns, solving Eqs. (16)–(24), we get the following results.

Case 1.

$$\begin{aligned} &A_1 = \mu = 0, \quad \lambda = \pm\frac{1}{2}\frac{pb}{\sqrt{R}(1+p)}, \\ &a = \frac{1}{2}\frac{B_1p^2b}{\sqrt{R}(1+p)}, \quad A_0 = \pm\sqrt{R}B_1, \\ &q = \pm 2\sqrt{R}bB_1 \end{aligned} \tag{25}$$

Case 2.

$$\begin{aligned} &A_0 = \pm\sqrt{R}B_1, \quad q = \pm 2\sqrt{R}bB_1, \\ &\lambda = \pm\frac{pb}{\sqrt{R}(1+p)}, \quad a = \pm 2\frac{B_1p^2b}{\sqrt{R}(1+p)}, \\ &A_1 = \pm\frac{\sqrt{R(\mu^2-1)}B_1}{R}. \end{aligned} \tag{26}$$

From (3), (13), (15) and (25)–(26), we obtain the following solutions for the nonlinear reaction-diffusion equation (11).

Family 1.

$$\begin{aligned} u_{11} = \Big\{ \pm\frac{q}{2b}\Big[1 \pm \tanh\big[\big(-p^2q/4a(1+p)\big)^{1/2} \\ \cdot \big(x \pm (-ab^2/q(1+p))^{1/2}t\big)\big]\Big]\Big\}^{-\frac{1}{p}}, \end{aligned} \tag{27}$$

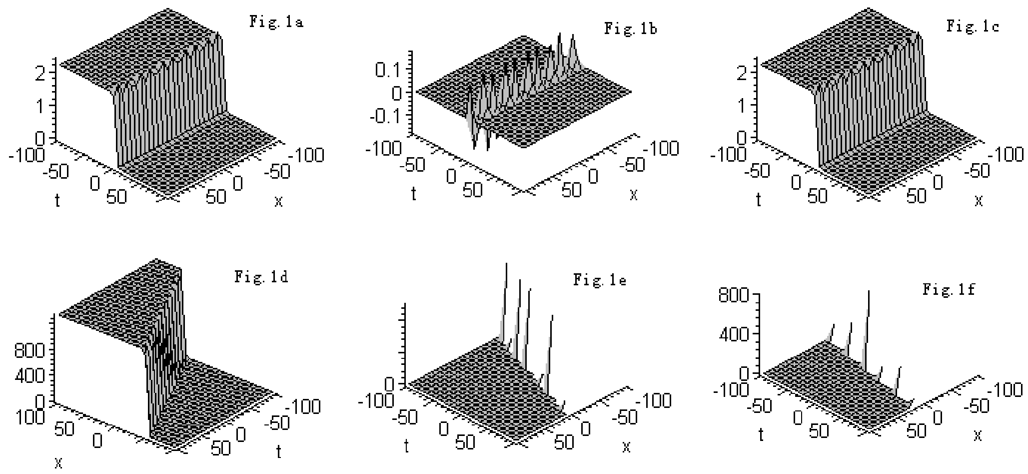

Fig. 1. Plots of solution (29) with $a=-2, q=11, b=1, \mu=2$ and different value of p: 1a)-1c) denote the real part, imaginary part and the modulus with $p=-3$; 1d) $p=-\frac{1}{3}$; 1e) $p=\frac{1}{3}$; 1f) $p=3$.

$$u_{12}=\Big\{\pm\frac{q}{2b}\Big[1\pm\coth\big[\big(-p^2q/4a(1+p)\big)^{1/2} \cdot \big(x\pm(-ab^2/q(1+p))^{1/2}t\big)\big]\Big]\Big\}^{-\frac{1}{p}}, \tag{28}$$

Family 2.

$$u_{21}=\Big\{\pm\frac{q}{2b}\Big[1\pm\sqrt{\mu^2-1}\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1} \pm\frac{\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}\Big]\Big\}^{-\frac{1}{p}}, \tag{29}$$

$$u_{22}=\Big\{\pm\frac{q}{2b}\Big[1\pm\sqrt{\mu^2-1}\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} \pm\frac{\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}\Big]\Big\}^{-\frac{1}{p}}, \tag{30}$$

where $\xi=x-\lambda t$, $\lambda=\pm\frac{bp}{\sqrt{R}(1+p)}$, $R=-\frac{qp^2}{a(1+p)}$.

Remark 1: It is easy to see that the solutions (27)–(28) reproduce the solutions (4.8)–(4.9) in [18]; if we set $\mu=0$, the solutions (29)–(30) coincide with the solutions (4.10)–(4.11) obtained in [18]. When $\mu\neq 0$, to our knowledge, the solutions (29)–(30) of (11) have not been found before.

Six plots are given to illustrate the properties of the new families of solutions (29) with various parameters. In Fig. 1, we consider solutions with the positive signs and different value of p, i. e., $p=-3,-\frac{1}{3},\frac{1}{3},3$.

3.2. The High-order Modified Boussinesq Equation

The high-order modified Boussinesq equation reads [18, 19],

$$u_{tt}+\alpha u_{xxt}+\beta u_{xxxx}+r(u^n)_{xx}=0, \tag{31}$$

where α, β, r and n are constants.

We make the transformation $u(x,t)=v(\xi)$, $\xi=x-\lambda t$. Then (31) becomes

$$\lambda^2 v''-\alpha\lambda v^{(3)}+\beta v^{(4)}+r(v^n)''=0. \tag{32}$$

Integrating (32) twice with respect to ξ, we obtain

$$\beta v''-\alpha\lambda v'+\lambda^2 v+rv^n=0, \tag{33}$$

with the integration constants taken to be zero.

Balancing v'' with v^n in (33), we get $m=\frac{2}{n-1}$. Therefore we make transformation

$$v(\xi)=\phi^{\frac{2}{n-1}}(\xi). \tag{34}$$

Then substituting (34) into (33) yields

$$\beta[2(3-n)\phi'^2+2(n-1)\phi\phi'']-2\alpha\lambda(n-1)\phi\phi' + (n-1)^2(\lambda^2\phi^2+r\phi^4)=0. \tag{35}$$

Now balancing ϕ'^2 (or $\phi\phi''$) with ϕ^4, we get $m=1$. So we assume that (35) has the following formal solutions

$$\phi(\xi)=A_0+A_1\sigma(\xi)+B_1\tau(\xi), \tag{36}$$

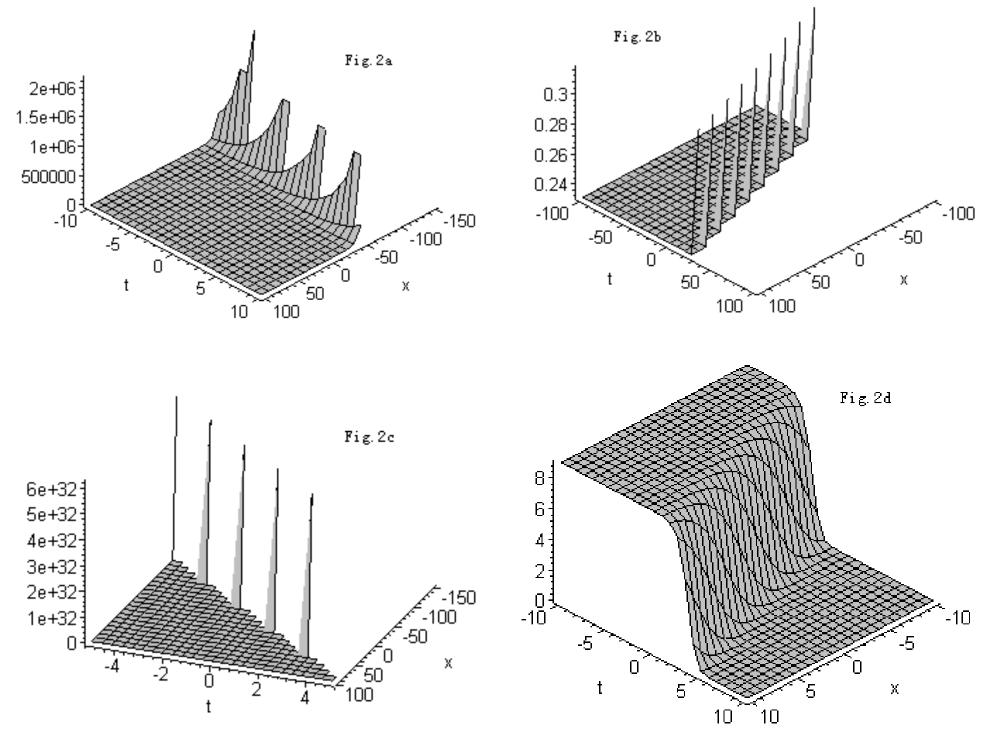

Fig. 2. Plots of the solution (44): 2a) n = –2, $\lambda=\mu=2$, $r=-10$, $\alpha=10$ and $\beta=-200$; 2b) $n=-\frac{1}{2}$, $\mu=2, r=-1, \alpha=4$ and $\beta=\frac{64}{25}$; 2c) $n=\frac{1}{2}$, $\lambda=20$, $\mu=2$, $r=-10$, $\alpha=10$ and $\beta=\frac{1200}{49}$; 2d) $n=2$, $\mu=2, r=-1, \alpha=4$ and $\beta=\frac{96}{25}$. It is necessary to point out the shaded areas in 2a, 2b and 2c refer to ∞ amplitude.

where $\sigma(\xi)$, $\tau(\xi)$ satisfy (1)–(2) and A_0, A_1, B_1 are constants to be determined later.

Proceeding as before, we can deduce a system of over-determined algebraic polynomials of $\{A_0, A_1, B_1, \mu, \lambda\}$. For simplicity, we don't list them in the paper. Solving the set, we obtain the following results.

Case 1.

$$\mu=A_0=B_1=\alpha=0, \quad r=-\frac{1}{2}\frac{(n+1)\lambda^2}{R^2{A_1}^2},$$

$$\beta=-\frac{1}{4}\frac{(1+n^2-2n)\lambda^2}{R}. \tag{37}$$

Case 2.

$$A_1=\mu=0, \quad r=-\frac{1}{4}\frac{\lambda^2}{R{B_1}^2},$$

$$\beta=\frac{1}{8}\frac{(1+n^2-2n)\lambda^2}{(n+1)R}, \; \alpha=\frac{1}{4}\frac{(n^2+2n-3)\lambda}{(n+1)\sqrt{R}},$$

$$A_0=\sqrt{R}B_1. \tag{38}$$

Case 3.

$$r=-\frac{1}{4}\frac{\lambda^2}{R{B_1}^2}, \quad A_1=\pm\frac{\sqrt{R(\mu^2-1)}B_1}{R},$$

$$\alpha=-\pm\frac{1}{2}\frac{(n^2+2n-3)\lambda}{\sqrt{R}(n+1)},$$

$$\beta=\frac{1}{2}\frac{(1-2n+n^2)\lambda^2}{(n+1)R}, \quad A_0=\pm\sqrt{R}B_1. \tag{39}$$

Therefore from (3), (34), (36) and (37)–(39), the high-order modified Boussinesq equation (31) has the following solutions.

Family 1. Using (37), the high-order modified Boussinesq equation (31) with $\alpha=0$ has the following solutions,

$$u_{11}=\left\{\pm\left(-(1+n)\lambda^2/2r\right)^{1/2} \tag{40}$$

$$\cdot\,\mathrm{sech}\left[\left(-(-1+n)^2\lambda^2/4\beta\right)^{1/2}(x-\lambda t)\right]\right\}^{\frac{2}{n-1}},$$

$$u_{12} = \left\{ \pm \left((1+n)\lambda^2/2r\right)^{1/2} \cdot \operatorname{csch}\left[\left(-(-1+n)^2\lambda^2/4\beta\right)^{1/2}(x-\lambda t)\right]\right\}^{\frac{2}{n-1}}. \quad (41)$$

Family 2. Using (38), Eq. (31) has the following solutions,

$$u_{21} = \left\{ \pm \left(-\lambda^2/4r\right)^{1/2}\left[1 \pm \tanh\left[(n-1)(n+3)\lambda \cdot \left(4(n+1)\alpha\right)^{-1}(x-\lambda t)\right]\right]\right\}^{\frac{2}{n-1}}, \quad (42)$$

$$u_{22} = \left\{ \pm \left(-\lambda^2/4r\right)^{1/2}\left[1 \pm \coth\left[\left((n-1)(n+3)\lambda \cdot \left(4(n+1)\alpha\right)^{-1}(x-\lambda t)\right]\right]\right\}^{\frac{2}{n-1}}. \quad (43)$$

Family 3. Using (39), Eq. (31) yields,

$$u_{11} = \left\{ \pm \sqrt{-\lambda^2/4r}\left[1 \pm \sqrt{\mu^2-1}\frac{\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1} \pm \frac{\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}\right]\right\}^{\frac{2}{n-1}}, \quad (44)$$

$$u_{12} = \left\{ \pm \sqrt{-\lambda^2/4r}\left[1 \pm \sqrt{\mu^2-1}\frac{\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1} \pm \frac{\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}\right]\right\}^{\frac{2}{n-1}}, \quad (45)$$

where λ, μ are arbitrary constants, $\xi = x-\lambda t$, $R = \frac{1}{4}\frac{(n^2+2n-3)^2\lambda^2}{\alpha^2(n+1)^2}$ and α, β, n satisfy $\beta = 2\frac{(n+1)\alpha^2}{n^2+6n+9}$.

Remark 2: From our results, the solutions (3.20)–(3.21), (3.24)–(3.25) obtained in [18] can be reproduced by (40)–(43); if we set $\mu = 0$ in (44)–(45), the solutions (3.22)–(3.23) obtained in [18] are also recovered. The other solutions, to our knowledge, have not been reported before.

Plots of the solution (44) (taking the positive signs) with some parameters are given in Figures 2.

3.3. The Variant Boussinesq Equations

Consider the variant Boussinesq equations [20–22]

$$u_t + v_x + uu_x + pu_{xxt} = 0, \quad (46)$$

$$v_t + (uv)_x + qu_{xxx} = 0. \quad (47)$$

Let $u(x,t) = U(\xi), v(x,t) = V(\xi), \xi = x+\lambda t$, then Eqs. (46) and (47) become

$$\lambda U' + V' + UU' + \lambda pU''' = 0, \quad (48)$$

$$\lambda V' + (UV)' + qU''' = 0. \quad (49)$$

Balancing U''' with UU', and U''' with $(UV)'$ leads to the following ansatz:

$$U = a_0 + a_1\sigma + a_2\sigma^2 + b_1\tau + b_2\sigma\tau, \quad (50)$$

$$V = A_0 + A_1\sigma + A_2\sigma^2 + B_1\tau + B_2\sigma\tau. \quad (51)$$

Proceeding as before, we can deduce a system of over-determined algebraic polynomials of $\{a_0, a_1, a_2, b_1, b_2, A_0, A_1, A_2, B_1, B_2, \mu, \lambda\}$. For simplicity, we don't list them in the paper. Solving the system, we obtain the following results.

Case 1.

$$a_2 = b_1 = b_2 = A_2 = B_1 = B_2 = 0, \quad \mu = \pm 1,$$

$$a_0 = \pm\frac{1}{6}\frac{18pq + a_1{}^2 + a_1{}^2Rp}{a_1p},$$

$$A_0 = \frac{1}{2}\frac{q(18q - a_1{}^2R)}{a_1{}^2},\ A_1 = \mp 3q,\ \lambda = \mp\frac{1}{6}\frac{a_1}{p}. \quad (52)$$

Case 2.

$$\mu = a_1 = b_1 = A_1 = B_1 = 0,$$

$$a_0 = \pm\frac{1}{6}\frac{-b_2{}^2R^2p - b_2{}^2R + 18pq}{b_2\sqrt{-R}p},$$

$$a_2 = \pm\frac{b_2}{\sqrt{-R}}, \quad A_0 = -\frac{1}{2}\frac{q(b_2{}^2R^2 + 18q)}{b_2{}^2R},$$

$$A_2 = 3\frac{q}{R},\ B_2 = \pm 3\frac{q\sqrt{-R}}{R},\ \lambda = \pm\frac{1}{6}\frac{b_2R}{p\sqrt{-R}}. \quad (53)$$

Case 3.

$$\mu = a_1 = b_1 = b_2 = A_1 = B_1 = B_2 = 0,$$

$$\lambda = \frac{1}{12}\frac{a_2R}{p},$$

$$a_0 = -\frac{1}{12}\frac{72pq + a_2{}^2R^2 + 4pa_2{}^2R^3}{a_2Rp},$$

$$A_2 = 6\frac{q}{R}, \quad A_0 = 2\frac{q(18q - a_2{}^2R^3)}{a_2{}^2R^2}. \quad (54)$$

Case 4.

$$B_1 = 0, \quad b_1 = 0, \quad A_2 = -3\frac{q(\mu^2-1)}{R},$$
$$A_1 = 3q\mu, \quad a_1 = \pm\frac{\sqrt{R(\mu^2-1)}b_2\mu}{\mu^2-1},$$
$$A_0 = \frac{1}{2}\frac{(18q\mu^2-18q-b_2{}^2R^2)q}{b_2{}^2R},$$
$$a_2 = \mp\frac{\sqrt{R(\mu^2-1)}b_2}{R}, \quad B_2 = \pm 3\frac{q(\mu^2-1)}{\sqrt{R(\mu^2-1)}},$$
$$\lambda = \pm\frac{1}{6}\frac{\sqrt{R(\mu^2-1)}b_2}{(\mu^2-1)p},$$
$$a_0 = \mp\frac{1}{6}\frac{18p\mu^2q-18pq+b_2{}^2R^2p+b_2{}^2R}{\sqrt{R(\mu^2-1)}b_2p}. \tag{55}$$

Therefore from (3), (50), (51) and (52)–(55), we can deduce the following 4 families of exact travelling wave solutions for the variant Boussinesq equations (46)–(47).

Family 1.

$$u_{11} = \pm\frac{1}{6}\frac{18pq+a_1{}^2+a_1{}^2Rp}{a_1p} + a_1\frac{R\,\mathrm{sech}\left[\sqrt{R}(x\pm\frac{a_1}{6p}t)\right]}{\mp\mathrm{sech}\left[\sqrt{R}(x\pm\frac{a_1}{6p}t)\right]+1},$$
$$v_{11} = \frac{1}{2}\frac{q(18q-a_1{}^2R)}{a_1{}^2} \pm 3q\frac{R\,\mathrm{sech}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]}{\mp\mathrm{sech}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]+1}. \tag{56}$$
$$u_{12} = \pm\frac{1}{6}\frac{18pq+a_1{}^2+a_1{}^2Rp}{a_1p} + a_1\frac{R\,\mathrm{csch}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]}{\mp\mathrm{csch}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]+1},$$
$$v_{12} = \frac{1}{2}\frac{q\left(18q-a_1{}^2R\right)}{a_1{}^2} \pm 3q\frac{R\,\mathrm{csch}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]}{\mp\mathrm{csch}\left[\sqrt{R}\left(x\pm\frac{a_1}{6p}t\right)\right]+1}. \tag{57}$$

Family 2.

$$u_{21} = \pm\frac{1}{6}\frac{-b_2{}^2R^2p-b_2{}^2R+18pq}{b_2\sqrt{-R}p} + b_2R^{\frac{3}{2}}[\pm\mathrm{i}\,\mathrm{sech}^2(\sqrt{R}\xi)+\mathrm{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)],$$
$$v_{21} = -\frac{1}{2}\frac{q\left(b_2{}^2R^2+18q\right)}{b_2{}^2R} + 3qR[\pm\mathrm{sech}^2(\sqrt{R}\xi)\pm\mathrm{i}\,\mathrm{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)], \tag{58}$$
$$u_{22} = \pm\frac{1}{6}\frac{-b_2{}^2R^2p-b_2{}^2R+18pq}{b_2\sqrt{-R}p} + b_2R^{\frac{3}{2}}[\pm\mathrm{csch}^2(\sqrt{R}\xi)+\mathrm{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)],$$
$$v_{22} = -\frac{1}{2}\frac{q(b_2{}^2R^2+18)}{b_2{}^2R} + 3qR[\pm\mathrm{csch}^2(\sqrt{R}\xi)\pm\mathrm{i}\,\mathrm{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)], \tag{59}$$

where $\xi = x \pm \frac{1}{6}\frac{b_2R}{p\sqrt{-R}}t$.

Family 3.

$$u_{31} = -\frac{1}{12}\frac{72pq+a_2{}^2R^2+4pa_2{}^2R^3}{a_2Rp} + a_2R^2\mathrm{sech}^2\left[\sqrt{R}\left(x+\frac{1}{12}\frac{a_2R}{p}t\right)\right],$$
$$v_{31} = 2\frac{q\left(18q-a_2{}^2R^3\right)}{a_2{}^2R^2} + 6qR\,\mathrm{sech}^2\left[\sqrt{R}\left(x+\frac{1}{12}\frac{a_2R}{p}t\right)\right]. \tag{60}$$
$$u_{32} = -\frac{1}{12}\frac{72pq+a_2{}^2R^2+4pa_2{}^2R^3}{a_2Rp} + a_2R^2\mathrm{csch}^2\left[\sqrt{R}\left(x+\frac{1}{12}\frac{a_2R}{p}t\right)\right],$$
$$v_{32} = 2\frac{q\left(18q-a_2{}^2R^3\right)}{a_2{}^2R^2} + 6qR\,\mathrm{csch}^2\left[\sqrt{R}\left(x+\frac{1}{12}\frac{a_2R}{p}t\right)\right]. \tag{61}$$

Family 4.

$$u_{41} = \mp\frac{1}{6}\frac{18p\mu^2q-18pq+b_2{}^2R^2p+b_2{}^2R}{b_2\sqrt{R(\mu^2-1)}p} \pm \frac{b_2\sqrt{R(\mu^2-1)}\mu R\,\mathrm{sech}(\sqrt{R}\xi)}{(\mu^2-1)(\mu\,\mathrm{sech}(\sqrt{R}\xi)+1)}$$

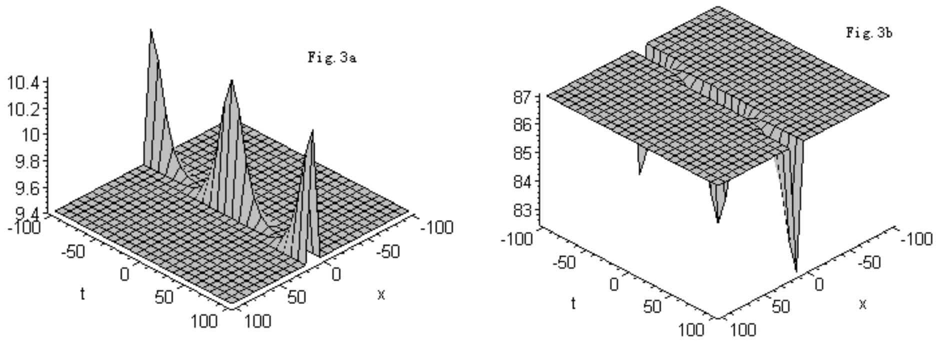

Fig. 3. Plots of the solutions (56) with $R=2, q=3, a_1=1, p=2$. Fig. 3a is the plot of u_{11} and Fig. 3b is the plot of v_{11}.

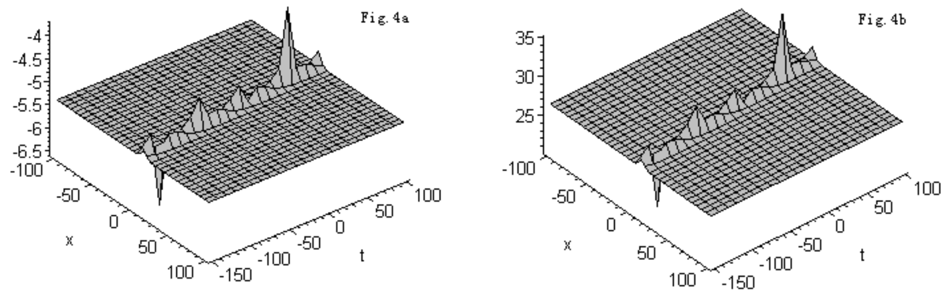

Fig. 4. Plots of the solution (62) with $p=q=R=b_2=1, \mu=2$. Fig. 4a is the plot of u_{41} and Fig. 4b is the plot of v_{41}.

$$\mp \frac{R\sqrt{R(\mu^2-1)}b_2(\operatorname{sech}(\sqrt{R}\xi))^2}{(\mu\operatorname{sech}(\sqrt{R}\xi)+1)^2}$$
$$+ \frac{b_2R^{3/2}\operatorname{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)}{(\mu\operatorname{sech}(\sqrt{R}\xi)+1)(\mu\tanh(\sqrt{R}\xi)+1)},$$
$$v_{41} = \frac{1}{2}\frac{(18q\mu^2-18q-b_2{}^2R^2)q}{b_2{}^2R} + 3\frac{q\mu R\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi)+1}$$
$$-3\frac{Rq(\mu^2-1)\left(\operatorname{sech}(\sqrt{R}\xi)\right)^2}{\left(\mu\operatorname{sech}(\sqrt{R}\xi)+1\right)^2} \tag{62}$$
$$\pm 3\frac{q(\mu^2-1)R^{3/2}\operatorname{sech}(\sqrt{R}\xi)\tanh(\sqrt{R}\xi)}{\sqrt{R(\mu^2-1)}(\mu\operatorname{sech}(\sqrt{R}\xi)+1)(\mu\tanh(\sqrt{R}\xi)+1)},$$
$$u_{42} = \mp\frac{1}{6}\frac{18p\mu^2q-18pq+b_2{}^2R^2p+b_2{}^2R}{b_2\sqrt{R(\mu^2-1)}p}$$
$$\pm \frac{b_2\sqrt{R(\mu^2-1)}\mu R\operatorname{csch}(\sqrt{R}\xi)}{(\mu^2-1)\left(\mu\operatorname{csch}(\sqrt{R}\xi)+1\right)}$$
$$\mp \frac{R\sqrt{R(\mu^2-1)}b_2\left(\operatorname{csch}(\sqrt{R}\xi)\right)^2}{\left(\mu\operatorname{csch}(\sqrt{R}\xi)+1\right)^2}$$
$$+ \frac{b_2R^{3/2}\operatorname{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)}{\left(\mu\operatorname{csch}(\sqrt{R}\xi)+1\right)\left(\mu\coth(\sqrt{R}\xi)+1\right)},$$
$$v_{42} = \frac{1}{2}\frac{(18q\mu^2-18q-b_2{}^2R^2)q}{b_2{}^2R} + 3\frac{q\mu R\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi)+1}$$
$$-3\frac{Rq(\mu^2-1)\left(\operatorname{csch}(\sqrt{R}\xi)\right)^2}{\left(\mu\operatorname{csch}(\sqrt{R}\xi)+1\right)^2} \tag{63}$$
$$\pm 3\frac{q(\mu^2-1)R^{3/2}\operatorname{csch}(\sqrt{R}\xi)\coth(\sqrt{R}\xi)}{\sqrt{R(\mu^2-1)}(\mu\operatorname{csch}(\sqrt{R}\xi)+1)(\mu\coth(\sqrt{R}\xi)+1)},$$

where $\xi = x+\lambda t$, $\lambda = \frac{1}{6}\frac{b_2\sqrt{R(\mu^2-1)}}{(\mu^2-1)p}$.

Remark 3: From our results, when setting $R=-b$, $c=\frac{1}{12}\frac{a_2R}{p}$, the solutions (60)–(61) are identical to those obtained by Fan [20] and Elwakil *et al.* [21]. The other solutions, to our knowledge, have not been found before.

Plots of solutions (56) and (62) with some parameters are given in Fig. 3 and Fig. 4, respectively.

4. Conclusions

In summary, making use of symbolic computation and the general projective Riccati equations method, we obtain many exact travelling wave solutions for a nonlinear reaction-diffusion equation, the high-order modified Boussinesq equation and the variant Boussinesq equation. From our results, we can not only recover the previous solutions obtained by some authors but also obtain some new and more general solitary wave solutions, singular solitary wave solutions and periodic solutions. It is shown that the key step of general projective Riccati equations method is to look for u as a polynomial in a variable which satisfies a system of two coupled Riccati equations. In fact, this method is one of the type which was subequation methods and we can choose other subequation, which must be defined in its canonical reduced form [26], such as a degenerate elliptic equation [27], a nongenerate elliptic equation [28], etc., as various subequations to search for more formal solutions. In forthcoming work, we search for other equation as subequatons to present the new method, which can be implemented in a computer algebraic system for finding new and more general formal solutions for other NPDEs and coupled NPDEs.

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